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CYCLIC THERMAL LOADING IN THE CREEP RANGE—I. LOW LEVELS OF THERMAL LOADING

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Abstract—Upper bounds on the deformation rates of structures subjected to cyclic thermal loading in the creep range are presented for low levels of thermal loading for a simple creep constitutive equation which includes the effect of thermal recovery. It is argued that the combination of material assumptions are conservative. The deformation rate is related to the strain rate in a reference material test at constant stress and cyclic temperature history. The level of stress is determined from the perfectly plastic shakedown solution and the reference temperature at a given instant is the same as that for the structure subjected to a constant load and the instantaneous temperature distribution.

1. INTRODUCTION

Methods of predicting the deformation of structures subjected to constant or steadily increasing load at both high and low temperatures are reasonably well understood. For cyclic loading, particularly thermal cycling, the problem becomes considerably more complex. At the present time there are particular solutions [1], but no general technique exists which is capable of giving an indication of the performance of a structure without recourse to a complete computer solution. As a result design codes tend to be rather conservative and the rules used do not reflect a deep understanding of structural performance under such loading conditions.

This paper attempts to provide such insight by the derivation of displacement bounds using a set of material and structural assumptions which attempt to be conservative for the range of materials and loading sequences which occur in practice.

In analysing the low temperature plastic behaviour of structures subjected to cyclic thermal loading Ponter [2] found that the most conservative estimate of the ratchet boundary was that given by the material model which allowed for hardening to a purely elastic state in regions of the structure where complete reversed plasticity occurred. We show in Section 3, through a simple example, that a similar conclusion operates within the creep regime and that the assumption of perfect plasticity may be nonconservative.

A further aspect of the material behaviour concerns the effects of material recovery during parts of the cycle when the stress is a minimum and the temperature is in the creep range. This circumstance frequently occurs in structures subject to temperature cycling within the creep range. Megahed, Ponter and Morrison [3, 4] have calculated the deformation rate of a simple two bar structure under these circumstances using two material models both of which gave correct steady state creep and rapid plastic loading properties, and one of which includes recovery effects. Of these material models the recovery model [5] gave the more conservative results. When compared with the results of experiments on both copper [3] and 316 SS [4] the recovery model, in fact, gave the best prediction, while a nonlinear viscous/strain-hardening model underestimated the deformation rate by about a factor of two, for the particular circumstance of the tests. For both sets of tests the stresses always remained positive and the effects of reverse cycling were not investigated.

O'Donnell and Porowski [1] have used an elastic/perfectly plastic/nonlinear viscous model to analyse the behaviour of a plate subjected to a constant axial load and a temperature distribution that oscillates between the values shown in Fig. 3, where the lower temperature was outside the creep range. Although this circumstance differs from the one discussed here it does appear that their material model does not properly take into account the cyclic hardening behaviour of the material nor the possibility of recovery of the flow stress during the high temperature part of the cycle. The results of Ponter [2] and Megahed, Ponter and Morrison [3, 4] show that the inclusion of both these effects can lead to more severe results.

In this paper we use a result due to Ponter [6] to calculate upper bounds on the deformation rate for the recovery model for a number of thermal loading problems in terms of the shakedown solution to the problem. We limit ourselves to analysing the class of structures that are subjected to constant primary load and cyclic thermal loading within the creep range. The thermal loading arises from a temperature distribution cycled between two prescribed limits. This circumstance corresponds to the loading experienced by components of the liquid sodium cooled fast nuclear reactor.

The bound can give reasonable values for the deformation rate provided the mechanism of incremental collapse associated with the shakedown solution is such that all the deformation occurs at one instant during the cycle at each point in the structure, i.e. no cyclic plastic deformation occurs. If it does, then the bound can severely overestimate the rate of deformation of the structure and then cease to be of any practical use in design. The reason for this severe overestimation is that the bound is expressed in terms of total energy dissipation over a stress cycle and does not distinguish between the part of the energy dissipation that results from the net increase in strain deformation of an element during a cycle and that which results from the reverse cyclic deformation of the element.

Using the bound, either the average or point deformation of the structure may be bounded in terms of the results of a reference material test where the stress level is given in terms of the perfectly plastic shakedown solution and where the reference temperature history may be expressed in terms of a nondimensional group of material properties, and the extremes of temperature during the cycle.

In the next section the shakedown properties of three simple structures are discussed in terms of the upper bound shakedown theorem. This is followed by a description of the recovery model and its associated bound. The bound is then applied to the three structures using the results of the shakedown analysis.

In an accompanying paper [7] the result is extended to problems involving very large temperature changes where, in the plasticity solution, regions of the structure suffer reverse plasticity.

2. STRUCTURAL BEHAVIOUR IN THE ABSENCE OF CREEP

Consider, for simplicity, the two bar structure shown in Fig. (1), where two bars of equal length and of cross-sectional area A and $4A$ are restrained to suffer equal exten-

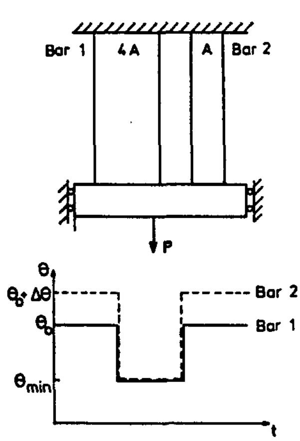

Fig. 1. Two bar structure subjected to a constant load P and cyclic variation of temperature.

sion. The loading is provided by a constant load P and a variation in temperature of the thinner bar between temperatures θ_{min} and $\theta_0 + \Delta\theta$ while the temperature of the thicker bar is varied between θ_{min} and θ_0. For an elastic perfectly plastic material with constant yield stress σ_y, elastic modulus E and linear coefficient of expansion α, the shakedown boundary is given by

$$\sigma_y = \frac{P}{P_L}\sigma_y + B\,\sigma_t, \quad B = \frac{1}{5} \tag{1}$$

$$\sigma_y = \frac{\sigma_t}{2} \tag{2}$$

where $P_L = 5A\sigma_y$, the limit load for P acting alone and $\sigma_t = \frac{4}{5}E\alpha\Delta\theta$, the maximum thermo-elastic stress, which occurs in Bar 2.

These boundaries are shown as AB and BC in the "Bree" diagram shown in Fig. (2). Throughout this paper σ_t will denote the maximum effective thermo-elastic stress resulting from the temperature cycle and, from dimensional arguments, has the form $\sigma_t = c\,E\alpha\Delta\theta$ where c is a numerical value less than unity.

Along boundary AB, there exists a stress history of the form $\sigma_i^* = \sigma_i^{*P} + \hat{\sigma}_i^{\theta}(t)$ where σ_i^{*P} is a stress field in equilibrium with the primary load P and $\hat{\sigma}_i^{\theta}(t)$ is the thermo-elastic stress field ($i = 1$ and 2) so that $\sigma_i^* \leq \sigma_y$, equality occurs at some time t_0 in the cycle. The mechanism of failure beyond AB is incremental collapse. The form of eqn (1) may be shown to occur in all structures provided the mechanism of plastic collapse does not change for the section of the shakedown limit for $0 \leq \sigma_t \leq 2\sigma_y$, i.e. AB in Fig. (2). This conclusion is implied by the upper bound shakedown theorem [2] which is discussed briefly in Appendix A. For most practical situations it appears that $0 \leq B \leq \frac{1}{4}$.

In Section 4 it is shown how the stress distributions found from shakedown analyses can be used to help bound the deformation rate in the creep range. The results are presented in terms of a reference test conducted at a reference stress σ_R and a reference

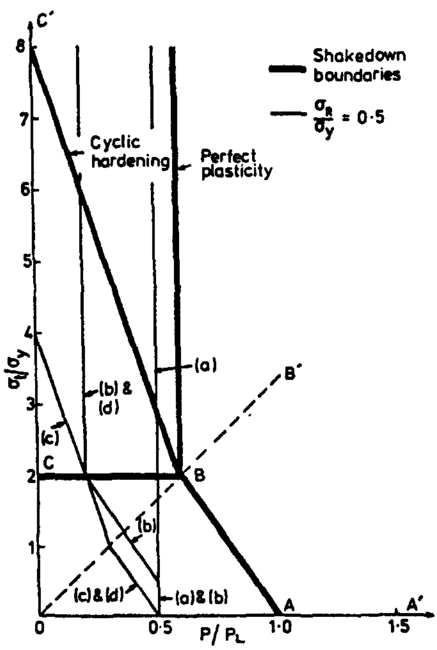

Fig. 2. Shakedown boundaries for two bar structure of Fig. 1, the thin lines represent contours of $\frac{\sigma_R}{\sigma_y} = 0.5$ for: (a) nonlinear viscous material model; (b) nonlinear viscous/perfectly plastic material model; (c) recovery model; (d) recovery model with limiting yield stress.

temperature history. In the region A′AOBB′ of the Bree diagram the reference stress is given by eqn (1) with σ_y replaced by σ_R:

$$\sigma_R = P\frac{\sigma_y}{P_L} + B\sigma_t \tag{3}$$

when σ_t is reduced to zero the form of σ_R reduces to the result originally discussed by Sim [8] and recently reviewed by Leckie [9] for constant load.

The calculation of a reference temperature is facilitated by constructing shakedown solutions where the yield stress varies spacially. We find that for the examples of this paper the form of the shakedown boundary, eqn (1), is retained because the mechanism of plastic incremental collapse remains the same as in the case of constant yield stress.

3. MATERIAL AND STRUCTURAL BEHAVIOUR IN THE CREEP RANGE

In Appendix B the response of the two bar structure of Fig. 1 to cyclic thermal loading is analysed for a material that creeps. Four different material models are considered. They are:

(a) Nonlinear viscous material.
(b) Nonlinear viscous/perfectly plastic material.
(c) Isotropic hardening/recovery model.
(d) Isotropic hardening/recovery model with limiting yield stress.

For simplicity the following assumptions concerning the rate of cycling and material properties are made:

(i) Rapid cycling. That is when the cycle time is small compared with characteristic material times.
(ii) θ_{min} is outside of the creep range.
(iii) The creep properties over the temperature range θ_0 to $\theta_0 + \Delta\theta$ are independent of temperature and can be expressed in the form:

$$\frac{\dot{\epsilon}}{\dot{\epsilon}_0} = \left[\frac{\sigma}{\sigma_0}\right]^n$$

where $\dot{\epsilon}_0$ is the strain-rate at a constant stress, σ_0.

For this particular problem the reference test is defined such that the strain-rate in the uniaxial specimen is the same as that experienced by the two bars. Contours of constant reference stress predicted by the four material models are shown in Fig. 2. Over all ranges of loading the recovery model gives the most conservative result. In the region A′AOBB′ of the Bree diagram the reference stress for the recovery model is given by eqn (3).

This simple example serves to illustrate the conclusions of Megahed, Ponter and Morrison [3, 4] that the recovery model is the most appropriate one for thermal loading problems. We can now proceed and examine this material model in more detail along with its predictions of structural behaviour.

The response of an element of material to a given multiaxial stress state can be expressed in terms of the state variable, s, that is a measure of the present size of the yield surface in stress space:

$$\dot{\epsilon}_{ij} = f(\emptyset - s)\frac{\partial\emptyset}{\partial\sigma_{ij}} \tag{4}$$

$$\dot{s} = h(s)\,f(\emptyset - s) - r(s) \tag{5}$$

where $\emptyset(\sigma_{ij})$ is a homogeneous function of degree one in σ_{ij}. In its simplest form f is given by the step function

$$f(\emptyset - s)\begin{cases} > 0 & \text{when } \emptyset = s, \dot{\emptyset} \geqslant 0 \\ = 0 & \text{when } \emptyset < s, \text{ or } \emptyset = s \text{ and } \dot{\emptyset} < 0. \end{cases}$$

The condition $\emptyset > s$ cannot be achieved.

The quantities $h(s)$ and $r(s)$ are the rates of strain hardening and thermal softening and depend upon the instantaneous value of the state variable s and the temperature θ.

For fast loading, $r(s)$ in eqn (5) can be ignored, then substitution of eqn (5) into eqn (4) gives

$$\dot{\epsilon}_{ij} = \frac{\dot{s}}{h(s)} \frac{\partial \emptyset}{\partial \sigma_{ij}} \tag{6}$$

and $h(s)$ can be determined from the shape of the monotonic stress–strain curve. During steady state conditions $\dot{s} = 0$, so

$$f(\emptyset - s) = \frac{r(s)}{h(s)} = \frac{r(\emptyset)}{h(\emptyset)}$$

and

$$\dot{\epsilon}_{ij} = \frac{r(\emptyset)}{h(\emptyset)} \frac{\partial \emptyset}{\partial \sigma_{ij}}. \tag{7}$$

The steady state creep behaviour can usually be described by an equation of the form

$$\dot{\epsilon}_{ij} = \frac{\dot{\epsilon}_0}{\sigma_0^n} \emptyset^n \frac{\partial \emptyset}{\partial \sigma_{ij}} \exp. - \frac{\Delta H}{R}\left(\frac{1}{\theta} - \frac{1}{\theta_0}\right) \tag{8}$$

where ΔH is the activation energy for creep, R is the universal gas constant, and $\dot{\epsilon}_0$ is the strain rate in a uniaxial test conducted at a constant stress σ_0 and constant temperature θ_0. The use of eqns (7) and (8) allow $r(s)$ to be determined. If the temperature range of interest is short then it is helpful to write eqn (8) in the form

$$\dot{\epsilon}_{ij} = \frac{\dot{\epsilon}_0}{\sigma_0^n} \emptyset^n \frac{\partial \emptyset}{\partial \sigma_{ij}} \exp\left[\gamma(\theta - \theta_0)\right] \tag{9}$$

$$\text{where} \quad \gamma = \frac{\Delta H}{R\theta_0^2}.$$

Using the above equations Ponter [6] has derived an upper bound on the displacement rate of a structure subjected to cyclic loading. If we limit ourselves to the cyclic state, assuming that one is reached [10, 11], then for the present problem the bound takes the form

$$\int_{S_T} T_i \dot{u}_i \, ds \leqslant \frac{\dot{\epsilon}_0}{\sigma_0^n} \int_V \emptyset^{n+1} \left[\frac{n}{n+1} \sigma_{ij}^* (t_0)\right] \left(\int_0^1 e^{\gamma(\theta - \theta_0)} \, d\tau\right) dV \tag{10}$$

where $\tau = \dfrac{t}{t_c}$, t_c being the cycle time,

$$\sigma_{ij}^* = \sigma_{ij}^{*P} + \sigma_{ij}^{*T} + \hat{\sigma}_{ij}^{\theta}(t) \tag{11}$$

σ_{ij}^{*P} is a stress distribution that is in equilibrium with the applied primary load P_i; σ_{ij}^{*T} is a stress distribution in equilibrium with a dummy load T_i applied in the direction of the required displacement rate; $\hat{\sigma}_{ij}^{\theta}(t)$ is the elastic stress distribution resulting from the variation of temperature through the structure at time t; t_0 is that time during the cycle when $\emptyset[\sigma_{ij}^{*}(t)]$ is a maximum; $\dot{u}_i$ is the mean displacement rate in the direction of T_i and S_T is the surface over which T_i is applied.

An optimum bound can be found by making appropriate choices of the magnitude of the load T_i and the stress distributions σ_{ij}^{*P} and σ_{ij}^{*T}. The optimum bound found in this way corresponds to the rapid cycle solution [12]. Rapid cycling is when the cycle time is such that the change in s due to recovery at $\emptyset = \emptyset^{\min}$ (the minimum value of $\emptyset$ during the cycle) is small compared to s. Then the rate of accumulation of strain in an element of material is determined by the maximum stress it experiences during the cycle.

Here we do not concern ourselves with finding the optimum bound. For $\sigma_{ij}^{*P} + \hat{\sigma}_{ij}^{\theta}(t_0)$ we simply use that equilibrium stress field that arises from the analysis of the shakedown boundary for an elastic perfectly plastic material. The stress field in equilibrium with T_i is taken from limit load calculations. Having used these stress distributions we then optimise the bound to find the best magnitude for T_i. A general result is obtained in the next section for the class of problems where the shakedown boundary is given by eqn (1). The bound is used to obtain solutions for a number of representative problems in Section 5.

4. GENERAL BOUND FOR THERMAL LOADING PROBLEMS

Ponter [2] has shown that for an elastic perfectly plastic material the ratchet mechanism in a wide range of thermal loading problems is the same as the collapse mechanism at constant temperature. Also, in the limited number of situations examined [2, 13], the ratchet mechanism is the same if the yield stress is a function of position in the structure. When these structures are subjected to cyclic thermal loading in the creep range a particularly useful form of the reference stress and reference temperature history results.

In this section structural behaviour when the creep properties are insensitive to temperature is analysed first and a reference stress is obtained. Temperature effects are then included and a reference temperature history is defined.

When a structure composed of an elastic/perfectly plastic material is subjected to mechanical and thermal loads corresponding to a point on the shakedown boundary,

$$\emptyset[\sigma_{ij}^{*P} + \hat{\sigma}_{ij}^{\theta}(t)] \leq \sigma_y \tag{12}$$

at each material point of the structure. Then for arbitrary values of P and σ_t an equilibrium stress field can be chosen such that

$$\emptyset[\sigma_{ij}^{*P} + \hat{\sigma}_{ij}^{\theta}(t_0)] \leq \sigma_R \tag{13}$$

where $$\sigma_R = P\left(\frac{\sigma_y}{P_L}\right) + B\,\sigma_t. \tag{14}$$

In the bound the magnitude of the dummy load T is unspecified. If we choose it such that

$$T = \eta\,\sigma_R\left(\frac{T_L}{\sigma_y}\right) \tag{15}$$

where T_L is the limit load for the structure, then an equilibrium stress field exists where

$$\emptyset(\sigma_{ij}^{*T}) \leq \eta\,\sigma_R. \tag{16}$$

Substituting eqns (13)–(16) in eqn (10) and noting that

$$\not{D}(\sigma_{ij}^{*}) \leq \not{D}(\sigma_{ij}^{*P} + \hat{\sigma}_{ij}^{\theta}(t)) + \not{D}(\sigma_{ij}^{*T})$$

we find, for a material whose creep properties are not a function of temperature, that

$$\dot{u}_i = \frac{u_i(\Delta t) - u_i(0)}{\Delta t} \leq \frac{\dot{\epsilon}_0}{n\eta\sigma_0^n}\left[\frac{n}{n+1}(1+\eta)\sigma_R\right]^n V\left(\frac{\sigma_y}{T_L}\right). \tag{17}$$

The optimum bound is when $\eta = \dfrac{1}{n}$, then

$$\dot{u}_i \leq \dot{\epsilon}_R\, V\frac{\sigma_y}{T_L} \tag{18}$$

where $\dot{\epsilon}_R$ is the strain rate at a stress σ_R given by eqn (14).

When calculating a reference temperature history we will consider the temperature cycle where

$$\theta = \theta_0 + w\Delta\theta \tag{19}$$

for a fraction λ of the cycle and $\theta = \theta_{min}$ throughout the structure during the rest of the cycle. The quantity w is a function of position in the structure. Use of the stress fields (13)–(16) in the bound now leads to the result

$$\dot{u}_i \leq \frac{\sigma_y}{T_L}\frac{\dot{\epsilon}_0}{\sigma_0^n}\sigma_R^n \int_V \{\lambda \exp(\gamma w\Delta\theta) + (1-\lambda)\exp[\gamma(\theta_{min} - \theta_0)]\}\,dV. \tag{20}$$

In a reference test conducted at constant stress σ_R whilst the temperature is cycled between θ_R (the reference temperature for the first part of the cycle) and θ_{min}, where

$$\theta_R = \theta_0 + x\Delta\theta, \tag{21}$$

the mean strain rate is

$$\dot{\epsilon}_R = \frac{\dot{\epsilon}_0}{\sigma_0^n}\sigma_R^n \{\exp(\gamma\, x\Delta\theta) + (1-\lambda)\exp[\gamma(\theta_{min} - \theta_0)]\}. \tag{22}$$

If the reference temperature for the first part of the cycle is defined as

$$V\exp(\gamma\, x\Delta\theta) = \int_V \exp(\gamma\, w\Delta\theta)\,dV \tag{23}$$

then eqn (20) reduces to eqn (18) with $\dot{\epsilon}_R$ given by eqn (22). It is sometimes possible to find a better reference temperature if the shakedown analysis is performed for a material whose yield stress is a function of position in the structure. For example if

$$\sigma_y^{nu} = \sigma_y\{\lambda\exp(\gamma\, w\Delta\theta) + (1-\lambda)\exp[\gamma(\theta_{min} - \theta_0)]\}^{-\frac{1}{n+1}} \tag{24}$$

$$= \sigma_y\, g(w) \tag{25}$$

then, provided the mechanism of failure is the same as in the constant yield stress situation, it can be shown using the upper bound shakedown theorem that the position of the shakedown boundary is given by

$$\sigma_y \overline{G} = P\left(\frac{\sigma_y}{P_L}\right) + B\,\sigma_t. \tag{26}$$

The quantity $\overline{G}$ is defined in Appendix A:

$$\overline{G} = \frac{\int_V \left\{\frac{\lambda \exp(\gamma\, w\Delta\theta)}{(d\epsilon^P)^{n+1}} + \frac{(1-\lambda)\exp[\gamma(\theta_{\min} - \theta_0)]}{(d\epsilon^P)^{n+1}}\right\}^{-\frac{1}{n+1}} dV}{\int_V d\epsilon^P\, dV} \tag{27}$$

where $d\epsilon^P$ is the effective strain increment at a material point that is compatible with the ratchet mechanism.

For the same material, under a constant load T, plastic collapse occurs when

$$T = \overline{G}\, T_L \tag{28}$$

(this equation is only valid if the mechanism of collapse under the load T is the same as the mechanism of failure at the shakedown boundary. This is true for the situations examined in this paper).

An equilibrium stress field can then be chosen where

$$\not{\!\beta}[\sigma_{ij}^{*P} + \hat{\sigma}_{ij}^{\theta}(t_0)] \leq \frac{\sigma_R g(w)}{\overline{G}} \tag{29}$$

and

$$\not{\!\beta}(\sigma_{ij}^{*T}) \leq \eta\, \frac{\sigma_R g(w)}{\overline{G}}. \tag{30}$$

Substitution of these two equations into the bound, eqn (10), and optimising w.r.t. η gives

$$\dot{u}_i \leq \frac{\dot{\epsilon}_0}{\sigma_0^n}\, \sigma_R^n \left(\frac{\sigma_y}{T_L}\right) V/\overline{G}^{n+1}. \tag{31}$$

This reduces to eqn (18) with $\dot{\epsilon}_R$ given by eqn (22) if the reference temperature for the first half of the cycle is defined using

$$\lambda \exp(\gamma\, x\Delta\theta) + (1-\lambda)\exp[\gamma(\theta_{\min} - \theta_0)] = \overline{G}^{-(n+1)}. \tag{32}$$

By making use of the Minkowski inequality [14]

$$\left\{\int (f+g)^{-\frac{1}{n+1}} dq\right\}^{-(n+1)} \geq \left[\int f^{-\frac{1}{n+1}} dq\right]^{-(n+1)} + \left[\int g^{-\frac{1}{n+1}} dq\right]^{-(n+1)} \tag{33}$$

eqn (32) becomes

$$\exp(\gamma\, x\Delta\theta) \geq G^{-(n+1)} \tag{34}$$

where

$$G = \int_V \left[\exp - \left(\frac{w\,\gamma\Delta\theta}{n+1}\right)\right] d\epsilon^P\, dV \Big/ \int_V d\epsilon^P\, dV \tag{35}$$

and is the ratio of limit loads when the yield stress is constant and when it varies with position in the structure (see eqn (A5) of Appendix A). The inequality of eqn (34) is in the wrong sense for the purposes of defining a bound, but it gives an indication of the form of the result and an approximation to the reference temperature history. If the deformation rate is given by eqn (18) then the reference temperature is strictly given by

$$\exp - \left(\frac{\gamma\, x\Delta\theta}{n+1}\right) \simeq G. \tag{36}$$

The best choice of reference temperature for the first part of the cycle is related through eqn (21) to the minimum value of x that results from the solution of eqns (23) and (36). These two equations for the reference temperature are exactly the same as those derived by Cocks and Ponter [15] for a structure subjected to constant load and the constant nonuniform temperature distribution of eqn (19). As pointed out in [15] the equation that gives the lowest value of x is the one that is derived from the stress distribution [either eqn (13) or eqn (29)] that most resembles that which occurs in practice.

The results of this section have a particularly simple interpretation: the net deformation rate of a structure subjected to a constant primary mechanical load and a cyclic history of temperature, can be found by utilising the results of a reference uniaxial creep test conducted at a constant stress σ_R and a cyclic temperature history. The definition of reference stress parallels that used for constant load and temperature [8, 9] and is related to the shakedown boundary through eqn (14). In the reference test the temperature at a given instant in the cycle is equal to the reference temperature when the structure experiences a constant temperature distribution identical to that which occurs instantaneously in the structure [15].

5. APPLICATION OF THE BOUND TO THERMAL LOADING PROBLEMS

In this section we concern ourselves with examining four particular thermal loading problems that have been shown by Ponter [2] to be representative of the types of problem encountered in high temperature design. These situations are shown diagrammatically in Figures 3, 6 and 9. The first problem is the classical Bree [16] situation, which is analysed in section 5.2, where a plate is subjected to a constant axial load and alternate cycles of uniform temperature and linear variation of temperature across the plate. In Section 5.2 we examine the problem of a simply supported plate carrying a uniform load subjected to the cycles of temperature shown in Fig. (6). Figure (9) shows a thin walled tube under a constant uniaxial load with a temperature front that oscillates over a portion of the tube. This problem is analysed in Section 5.3 for both short and long extents of travel.

The shakedown boundaries for each situation are shown in Figs. 4, 7, 10 and 11 for a material whose yield stress does not vary with temperature. For each situation the high primary load boundary can be represented by an equation of the form of eqn (1).

5.1. *The Bree plate*

The thermo-elastic stress distribution resulting from the nonuniform temperature distribution is shown in Fig. (3). The stress varies linearly across the plate between σ_t at $z = -d/2$ and $-\sigma_t$ at $z = d/2$, where

$$\sigma_t = \frac{E\alpha\Delta\theta}{2}.$$

The position of the shakedown boundary is given by [2, 16]

$$\sigma_y = \sigma_P + \frac{\sigma_t}{4} \tag{37}$$

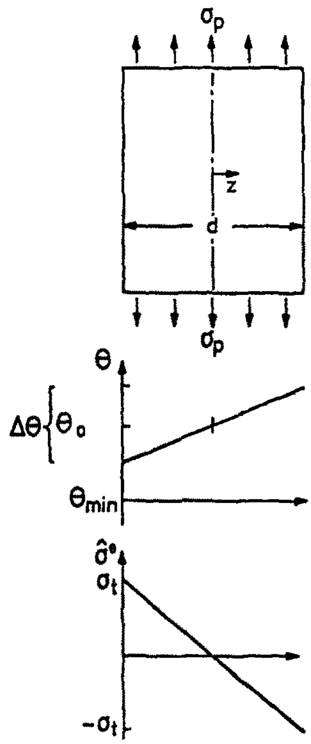

Fig. 3. Temperature and thermo-elastic stress distributions for the Bree plate.

when the yield stress is constant, and from eqn (14), the reference stress is

$$\sigma_R = \sigma_P + \frac{\sigma_t}{4}. \tag{38}$$

Contours of constant reference stress are plotted in Fig. (4).

Using eqn (23), the reference temperature for the nonuniform temperature distribution is

$$\exp \gamma\, x\Delta\theta = \int_{-1/2}^{1/2} \exp\left(\gamma\, \Delta\theta\, \frac{z}{d}\right) \cdot \mathrm{d}\left(\frac{z}{d}\right). \tag{39}$$

Integration of eqn (39) gives

$$x = \frac{1}{\gamma\, \Delta\theta} \ln\left[\frac{2}{\gamma\, \Delta\theta} \sinh \frac{\gamma\, \Delta\theta}{2}\right]. \tag{40}$$

For small values of $\gamma\, \Delta\theta$ this reduces to

$$x = \frac{\gamma\, \Delta\theta}{24} = \frac{n}{24} \cdot \frac{\gamma\, \Delta\theta}{n}. \tag{41}$$

Both these equations are plotted in Fig. (5).

For this particular problem a reference temperature history is best found using the shakedown solution when the yield stress is given by

$$\sigma_y^{nu} = \sigma_y \left\{\lambda \exp\left(\gamma\, \Delta\theta\, \frac{z}{d}\right) + (1 - \lambda) \exp\left[\gamma\, (\theta_{min} - \theta_0)\right]\right\}^{-1/n} \tag{42}$$

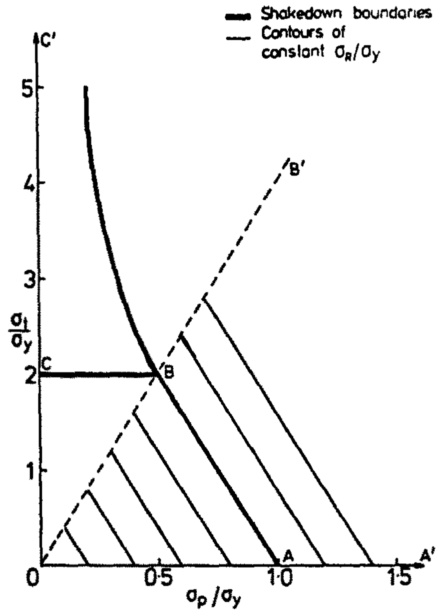

Fig. 4. Bree diagram showing contours of constant reference stress.

the result is given by eqn (32) with $(n + 1)$ replaced by n. $\overline{G}$ can be found using the upper bound shakedown boundary theorem and is given by eqn (27) with $d\epsilon^P$ set equal to unity and $(n + 1)$ replaced by n. The quantity G is given by eqn (35) with $d\epsilon^P$ again set equal to unity and $(n + 1)$ set equal to n. The reference temperature given by eqn (36) is then

$$\exp - \frac{\gamma\,\Delta\theta}{n} x = \int_{-1/2}^{1/2} \exp - \left(\frac{\gamma\,\Delta\theta}{n}\frac{z}{d}\right) \cdot \mathrm{d}\left(\frac{z}{d}\right). \tag{43}$$

Integration of eqn (43) gives

$$x = -\frac{n}{\gamma\,\Delta\theta} \ln\left[\frac{2n}{\gamma\,\Delta\theta} \sinh \frac{\gamma\,\Delta\theta}{2n}\right] \tag{44}$$

which for small values of $\frac{\gamma\,\Delta\theta}{n}$ becomes

$$x = -\frac{1}{24}\frac{\gamma\,\Delta\theta}{n}. \tag{45}$$

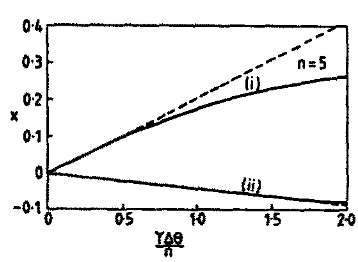

Fig. 5. Reference temperature, $\theta_R = \theta_0 + x\Delta\theta$, for Bree plate predicted by (i) eqn (40) and (ii) eqn (44). The dashed lines represent a linearisation of the results.

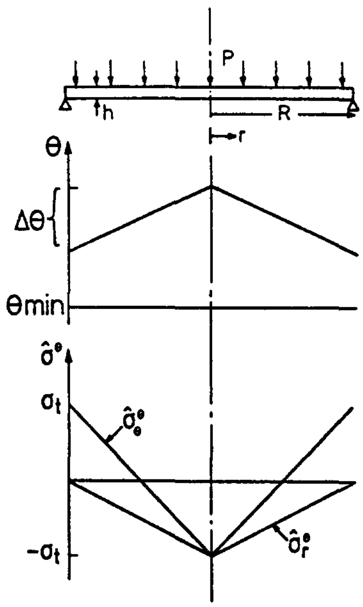

Fig. 6. Thermo-elastic stress distribution in simply supported plate for temperature distribution shown.

These two equations are also plotted in Fig. (5) where they can be compared with eqns (40) and (41). They give the better definition of reference temperature.

5.2. *Plate supporting a normal pressure*

Shakedown occurs in the plate of Fig. (6) when

$$\sigma_y = P\left(\frac{\sigma_y}{P_L}\right) + \frac{\sigma_t}{4} \tag{46}$$

where

$$P_L = \frac{3}{2}\sigma_y\frac{h^2}{R^2}$$

and the maximum thermo-elastic stress in the structure is

$$\sigma_t = \frac{E\alpha\Delta\theta}{3}.$$

Equation (46) is for a material that yields according to the Tresca criterion, and the results of the bound are strictly for a material whose creep rate is given by the associated Tresca flow rule. But provided the size of the surface of constant energy dissipation for a von Mises material is chosen such that Tresca and von Mises flow rules give the same creep rate in uniaxial tension the result presented below is also a bound for a von Mises material.

The mechanism of deformation associated with eqn (46) is a conical mode. Cocks [13] has shown that this is the correct mechanism provided

$$\sigma_t \leqslant \frac{12}{7}\sigma_y \tag{47}$$

when the yield stress is uniform throughout the structure. When the yield stress is a function of position in the structure the failure mechanism is the same, but the limiting

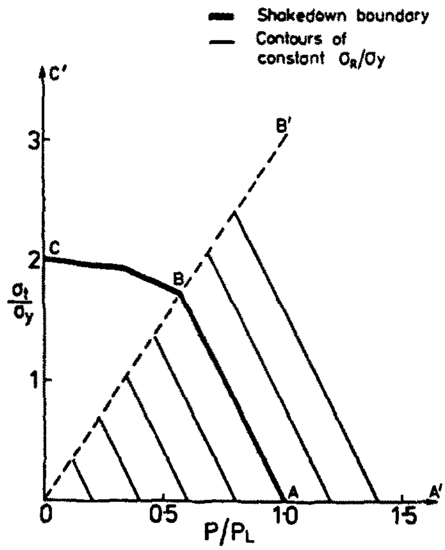

Fig. 7. Bree diagram for simply supported plate showing contours of constant reference stress.

value of σ_t is less than that given by eqn (47). Details of this limit are given by Cocks [13].

From eqn (14) the reference stress is given by

$$\sigma_R = P\left(\frac{\sigma_y}{P_L}\right) + \frac{\sigma_t}{4}. \tag{48}$$

Contours of constant reference stress are plotted in Fig. (7).

The reference temperature for the part of the cycle where the temperature distribution is nonuniform, [eqn (23)], is

$$\exp \gamma\, x\Delta\theta = \int_0^1 2\left\{\exp\left[\gamma\,\Delta\theta\left(1 - \frac{r}{R}\right)\right]\right\}\cdot\frac{r}{R}\cdot \mathrm{d}\left(\frac{r}{R}\right). \tag{49}$$

Integration of eqn (49) gives

$$x = 1 - \frac{1}{\gamma\,\Delta\theta}\ln\left[\frac{0.5}{\left(\frac{1}{\gamma\,\Delta\theta}\right)^2 - [\exp - \gamma\,\Delta\theta]\left[\left(\frac{1}{\gamma\,\Delta\theta}\right)^2 + \frac{1}{\gamma\,\Delta\theta}\right]}\right] \tag{50}$$

which for small values for $\gamma\,\Delta\theta$ reduces to

$$x = \frac{1}{3} + \frac{n+1}{36}\cdot\frac{\gamma\,\Delta\theta}{n+1}. \tag{51}$$

These two equations are plotted in Fig. (8).

Using the conical mode of failure G can be calculated using eqn (35) when the yield stress varies with position according to eqn (24):

$$G = \int_0^1 \exp - \left[\frac{\gamma\,\Delta\theta}{n+1}\left(1 - \frac{r}{R}\right)\right]\cdot \mathrm{d}\left(\frac{r}{R}\right). \tag{52}$$

Using eqn (36) the reference temperature for the first part of the cycle can be found:

$$\exp - x\frac{\gamma\,\Delta\theta}{n+1} = \int_0^1 \exp - \left[\frac{\gamma\,\Delta\theta}{n+1}\left(1-\frac{r}{R}\right)\right]\cdot \mathrm{d}\left(\frac{r}{R}\right) \tag{53}$$

which upon integration gives

$$x = \frac{n+1}{\gamma\,\Delta\theta}\ln\left[\frac{n+1}{\gamma\,\Delta\theta}\Big/\left(1-\exp-\frac{\gamma\,\Delta\theta}{n+1}\right)\right]. \tag{54}$$

For small values of $\frac{\gamma\,\Delta\theta}{n+1}$ eqn (54) reduces to

$$x = \frac{1}{2} - \frac{1}{24}\frac{\gamma\,\Delta\theta}{n+1}. \tag{55}$$

Both of these equations are plotted in Fig. (8) where they can be compared with eqns (50) and (51). For low values of the quantity $\frac{\gamma\,\Delta\theta}{n+1}$ eqns (50) and (51) give the better reference temperature. While for

$$\frac{\gamma\,\Delta\theta}{n+1} > \frac{12}{2n-1}$$

eqns (54) and (55) represent the better choice. The reason for this can be found by examining the two moment distributions used in the definition of reference temperature, and comparing them with the distribution obtained from an exact analysis. Venkatraman and Hodge [17] give the bending moment distribution for $\Delta\theta = 0$. The moments are greater near the centre of the plate, and less near the outer edge, than they are for a perfectly plastic material. As $\Delta\theta$ is increased the bending moments supported by the centre of the plate decreases, while their magnitude increases in the outer regions. The bending moment distribution is then similar to that for a perfectly plastic material whose yield stress is not a function of temperature. Use of this moment distribution in the bound leads to the better reference temperature. If $\Delta\theta$ is increased still further then the hoop bending moment at the centre of the plate drops below that at the outer edge, the bending moment distribution is more like that for a material with a yield stress that is a function of position, and use of this moment distribution in the bound leads to the better reference temperature.

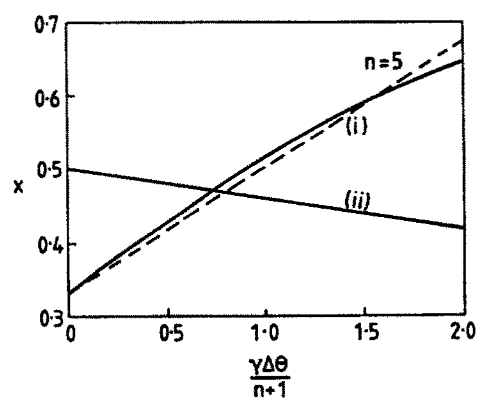

Fig. 8. Reference temperature, $\theta_R = \theta_0 + x\Delta\theta$, for simply supported plate predicted by (i) eqn (50) and (ii) eqn (54). The dashed line is a linearisation of the result.

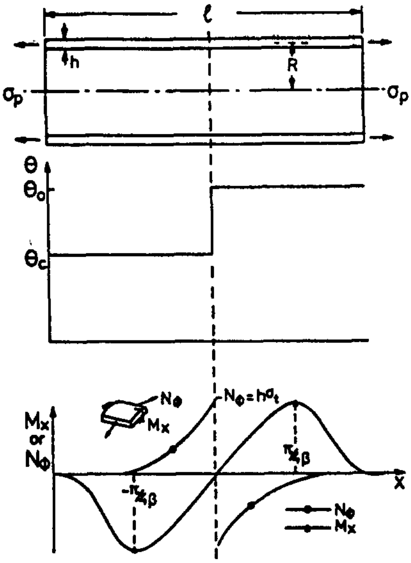

Fig. 9. Thermo-elastic stress distribution in the vicinity of a temperature front that moves along the tube.

5.3 *Temperature front moving along a tube*

The stress distribution in the vicinity of the temperature front is shown in Fig. (9), [18]. If the length of tube over which the temperature front moves is small compared with the characteristic decay length of the hoop component of stress then the problem is one of short travel. The element of material over which the front moves is subjected to a cyclic hoop component of stress

$$\Delta\sigma_0 = 2\sigma_t$$

where

$$\sigma_t = \frac{E\alpha\Delta\theta}{2}$$

and the cyclic thermal stress in the remainder of the tube is practically zero. Provided

$$\sigma_t \leqslant \frac{\sigma_p}{2} \qquad (56)$$

where σ_p is the axial stress applied to the tube, it is easy to show that for both von Mises and Tresca materials the shakedown boundary is given by

$$\sigma_y = \sigma_p.$$

So the reference stress is $\sigma_R = \sigma_p$ and the reference temperature can be defined using eqn (23). But if we just concentrate on that part of the tube that always experiences the higher temperature θ_0, then the deformation rate of this region is

$$\frac{\dot{u}_a}{l_h} \leqslant \frac{\dot{\epsilon}_0}{\sigma_0^n}\sigma_R^n \qquad (57)$$

and, for this particular problem, the temperature in the reference test should be maintained at $\theta = \theta_0$ throughout the duration of the test.

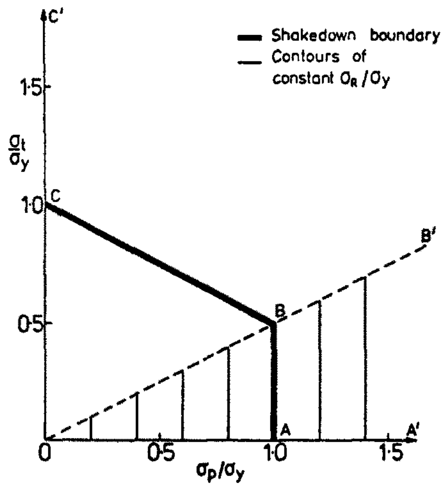

Fig. 10. Bree diagram for short travel problem showing contours of constant reference stress.

Contours of constant reference stress are shown in Fig. (10) up to the limit of eqn (56) which is represented by the line OBB'.

When the temperature front oscillates over a long length of tube each element of material experiences a thermo-elastic hoop stress that alternates between $\pm\ \sigma_t$. Because a significant portion of the tube experiences the same range of loading this hoop stress is unable to redistribute. For a Tresca material shakedown occurs if

$$\sigma_p + \sigma_t \leqslant \sigma_y. \tag{58}$$

If the reference stress is given by

$$\sigma_R = \sigma_p + \sigma_t \tag{59}$$

then, from eqn (20), the deformation rate is

$$\begin{aligned} \frac{\dot{u}_a}{l} &\leqslant \dot{\epsilon}_0\{\lambda + (1 - \lambda)\exp\left[\gamma\left(\theta_c - \theta_0\right)\right]\} \\ &\leqslant \dot{\epsilon}_R \end{aligned} \tag{60}$$

where l is the length of travel of the front, if the reference temperature history is chosen such that θ_0 is maintained for a fraction λ of the test and $\theta = \theta_c$ for the remainder. For

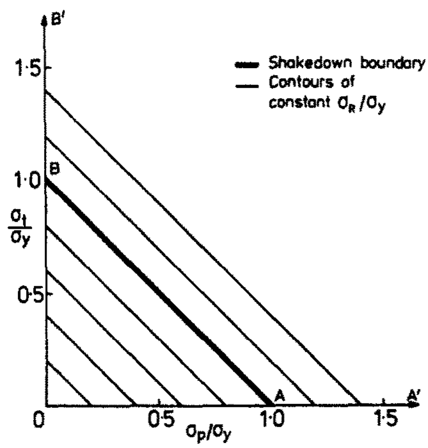

Fig. 11. Bree diagram for long travel problem showing contours of constant reference stress.

simplicity it has been assumed in this example that each element of material spends the same amount of time at each temperature. Contours of constant reference stress are plotted in Fig. (11). Unlike in the previous examples eqn (60) applies over the full range of loading situations. A consequence of this result is that the tube can deform axially under zero applied primary load.

6. CONCLUSIONS

In this paper it has been shown how the results from shakedown analyses can be used to bound the deformation rate of a structure in the creep range. The deformation rate is related to the strain rate in an uniaxial specimen subjected to a constant reference stress and cyclic temperature. The reference stress is given by

$$\sigma_R = P\left(\frac{\sigma_y}{P_L}\right) + B\,\sigma_t$$

where P_L is the limit load of the structure and B is the gradient of the shakedown boundary on a plot of P/P_L against σ_t/σ_y. The reference temperature at a given instant is the same as that for the structure subjected to a constant load and the instantaneous temperature distribution. For the situations examined here it is found that a conservative result is obtained if the temperature is cycled between θ_{mean}, the mean temperature when there is a temperature gradient through the structure, and θ_{min}, the uniform temperature in the structure.

A detailed comparison of the results given here with those given by O'Donnell and Porowski [1] for a viscous/perfectly plastic material is given in an accompanying paper [7]. In [7] the results developed here are extended to the region CC'BB' of the Bree diagram.

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APPENDIX A

Shakedown boundaries for cyclic thermal loading

Consider a body subjected to a constant load P. If $d\epsilon_{ij}^P$ is the strain field compatible with unit displacement in the direction of P at collapse, then from the principle of virtual work we find

$$P_L = \int_V \sigma_{ij}^y \, d\epsilon_{ij}^P \, dV \tag{A1}$$

where σ_{ij}^y is the stress on the yield surface associated with the plastic strain increment $d\epsilon_{ij}^P$. If $d\epsilon^P$ is the effective strain component at a point then

$$P_L = \sigma_y \int_V d\epsilon^P \, dV. \tag{A2}$$

If the yield stress is now a function of position in the structure, for example if

$$\sigma_y^{nu} = \sigma_y \exp - \left(\frac{w \, \gamma\Delta\theta}{n+1}\right) \tag{A3}$$

where w is defined in eqn (19), then, if the mechanism of collapse is the same, the limit load is given by

$$P_L^{nu} = \sigma_y \int_V \exp - \left(\frac{w \, \gamma\Delta\theta}{n+1}\right) \cdot d\epsilon^P \, dV \tag{A4}$$

or

$$P_L^{nu} = P_L \, G \tag{A5}$$

where

$$G = \int_V \exp - \left(\frac{w \, \gamma\Delta\theta}{n+1}\right) \cdot d\epsilon^P \, dV \bigg/ \int_V d\epsilon^P \, dV. \tag{A6}$$

Shakedown can be analysed in much the same way. The stress at a point in the body is given by

$$\sigma_{ij} = \hat{\sigma}_{ij}^P + \hat{\sigma}_{ij}^\theta(t) + \bar{\rho}_{ij}. \tag{A7}$$

If at a time t_0 the stress $\sigma_{ij} = \sigma_{ij}^y$, then applying the principle of virtual work, we find

$$\sigma_y \int_V d\epsilon^P \, dV = P + \sigma_t \int_V \alpha_{ij} \, d\epsilon_{ij}^P \, dV \tag{A8}$$

where $\sigma_t \, \alpha_{ij} = \hat{\sigma}_{ij}(t_0)$. This can be written in the form of eqn (1)

$$\sigma_y = P\left(\frac{\sigma_y}{P_L}\right) + B \, \sigma_t \tag{A9}$$

where P_L is given by eqn (A2) and

$$B = \int_V \alpha_{ij} \, d\epsilon_{ij}^P \, dV \bigg/ \int_V d\epsilon^P \, dV. \tag{A10}$$

If the mechanism changes for increasing $\Delta\theta$ it may be shown [2] that

$$\sigma_y \geq \frac{P}{P_L} \sigma_y + B \, \sigma_t.$$

When the yield stress is a function of position in the structure, i.e. when

$$\sigma_y^{nu} = \sigma_y \{\lambda \exp(\gamma \, w\Delta\theta) + (1-\lambda) \exp[\gamma(\theta_{min} - \theta_0)]\}^{-\frac{1}{n+1}} \tag{A11}$$

we find

$$\sigma_y \int_V \{\lambda \exp(\gamma w \Delta\theta) + (1-\lambda)\exp[\gamma(\theta_{min}-\theta_0)]\}^{-\frac{1}{n+1}} d\epsilon^P \, dV = P + \sigma_t \int_V \alpha_{ij} \, d\epsilon_{ij}^P \, dV \qquad (A12)$$

or

$$\bar{G}\,\sigma_y = P\left(\frac{\sigma_y}{P_L}\right) + B\,\sigma_t \qquad (A13)$$

where

$$\bar{G} = \frac{\int_V \left[\frac{\lambda \exp(\gamma w \Delta\theta)}{(d\epsilon^P)^{n+1}} + \frac{(1-\lambda)\exp\gamma(\theta_{min}-\theta_0)}{(d\epsilon^P)^{n+1}}\right]^{-\frac{1}{n+1}} dV}{\int_V d\epsilon^P \, dV}. \qquad (A14)$$

In Section 4 it is shown that the choice of reference temperature in the creeping problem depends on $\bar{G}^{-(n+1)}$, see eqn (32).

APPENDIX B

Effect of the choice of material model on the behaviour of a two bar structure

In this Appendix we analyse the response of the two bar structure of Fig. 1 to the loading situation indicated there for different assumptions of material behaviour. The material models considered are:

(a) Nonlinear viscous
(b) Nonlinear viscous/perfectly plastic
(c) Recovery
(d) Recovery model with limiting yield stress.

In each instance it is assumed that θ_{min} is outside of the creep range and that over the temperature range θ_0 to $\theta_0 + \Delta\theta$ the creep properties of the material are independent of temperature. Rapid cycle loading conditions are also assumed.

The results are presented in terms of a reference stress, σ_R, which is defined as the constant stress that results in the accumulation of a strain ϵ_R after a time t_R at a temperature θ_0, where ϵ_R is the strain accumulated in one of the bars during a cycle and t_R is the time spent in the creep range. Contours of constant reference stress are plotted on Fig. 2 for each assumption of material behaviour.

(a) *Nonlinear viscous material model.* For a nonlinear viscous material there is no accumulation of plastic strain at the low temperature. Strain only accumulates at the high temperature where, from compatibility considerations, the stress in the two bars must be the same. It is easy to show that

$$\frac{\sigma_R}{\sigma_y} = \frac{P}{P_L} \qquad (B1)$$

where $P_L = 5A\,\sigma_y$, in all instances.

(b) *Nonlinear viscous/perfectly plastic material model.* The result given by equation (A1) is valid for this material model provided the stresses in the two bars does not exceed the yield stress, σ_y, during any part of the cycle. Yield is violated first in bar 2 at $\theta = \theta_{min}$ when:

$$\frac{P}{P_L} + \frac{\sigma_t}{\sigma_y} = 1 \qquad (B2)$$

where $\sigma_t = \frac{4}{5}E\alpha\Delta\theta$.

For $\frac{P}{P_L} + \frac{\sigma_t}{\sigma_y} \geq 1$

the stress in bar 2 oscillates between σ_y at $\theta = \theta_{min}$ and $\sigma_y - \sigma_t$ at $\theta = \theta_0 + \Delta\theta$. From compatibility and equilibrium considerations it follows that:

$$\frac{\sigma_R}{\sigma_y} = \frac{5}{4}\frac{P}{P_L} + \frac{1}{4}\left(\frac{\sigma_t}{\sigma_y} - 1\right). \qquad (B3)$$

This result applies provided

$$\frac{\sigma_t}{\sigma_y} \leq 2$$

then reversed plastic strains occur in bar 2. Thermal strains resulting from any further increase in $\Delta\theta$ are matched by plastic strains with no increase of stress in bar 3, so:

$$\frac{\sigma_R}{\sigma_y} = \frac{5}{4}\frac{P}{P_L} + \frac{1}{4}. \tag{B4}$$

(c) *Recovery model.* The recovery model is described in section 3. Structural behaviour under conditions of rapid cycling is discussed in [6] and [7]. Under this loading condition the rate of accumulation of strain is determined by the maximum effective stress experienced during the cycle. This strain is a combination of creep and time independent plastic strain. For the present uniaxial situation if we let σ_R be the maximum stress in each bar during the cycle then,

$$\sigma_R = \sigma_{p1} + \frac{1}{4}\sigma_t = \sigma_{p2} \tag{B5}$$

where σ_{p1} and σ_{p2} are the stresses in bars 1 and 2 respectively when $\theta = \theta_{min}$. Then in order to satisfy equilibrium

$$\frac{\sigma_R}{\sigma_y} = \frac{P}{P_L} + \frac{1}{5}\frac{\sigma_t}{\sigma_y}. \tag{B6}$$

This is valid up to $\sigma_{p2} = \frac{\sigma_t}{2}$, i.e. until:

$$\frac{P}{P_L} = 0.3\frac{\sigma_t}{\sigma_y}.$$

Then the effective stress in bar 2 is the same at both ends of the cycle. Bar 2 now suffers reversed plastic strains (see eqn (3) or [7]) while the stress oscillates between $\pm\frac{\sigma_t}{2}$. From equilibrium consideration we calculate the reference stress to be:

$$\frac{\sigma_R}{\sigma_y} = \frac{5}{4}\frac{P}{P_L} + \frac{1}{8}\frac{\sigma_t}{\sigma_y}. \tag{B7}$$

(d) *Recovery model with limiting yield stress.* For this material model the results given by eqns (A6) and (A7) are valid until

$$\sigma_{p2} = \sigma_y$$

i.e.

$$\frac{\sigma_t}{2} = \sigma_y. \tag{A8}$$

Then any increase in thermal strain in bar 2 is offset by an increase in plastic strain with no increase of stress. The reference stress is then given by eqn (B4).